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A generalization of determinant formulae for the solutions of Painlevé II and XXXIV equations

Kenji Kajiwara† and Tetsu Masuda‡

† Department of Electrical Engineering, Doshisha University, Kyotanabe, Kyoto, 610-0321, Japan
‡ Department of Physics, Ritsumeikan University, Kusatsu, Shiga, 525-8577, Japan

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Abstract. A generalization of determinant formulae for the classical solutions of Painlevé XXXIV and Painlevé II equations are constructed using the technique of Darboux transformation and Hirota's bilinear formalism. It is shown that the solutions admit determinant formulae even for the transcendental case.

1. Introduction

It is well known that solutions for the Painlevé equations play a role of special functions in nonlinear science [1]. Originally, Painlevé derived these equations in order to find new transcendental functions determined by second-order ordinary differential equations possessing the so-called Painlevé property. Recently, irreducibility of solutions of the Painlevé equations has been proved by Umemura *et al* [2, 3]. Umemura first gave a rigorous definition of classical functions: starting from the field of rational functions, if a function is obtained by finite numbers of iterations of the following permissible operations:

- differentiation,
- arithmetic calculations,
- solving homogeneous linear ordinary differential equations,
- substitution into Abelian functions,

then that function is called 'classical'.

Umemura proved that solutions of the Painlevé equations are not classical in general in the above sense. However, it is known that they admit classical solutions for special values of parameters. One is the rational or algebraic solutions, and another is the transcendental classical solutions which are expressed by rational functions in various special functions and their derivatives.

In this paper, we discuss the Painlevé II equation (P_{II}),

$$\frac{d^2u}{dz^2} = 2u^3 - 4zu + 4\left(\alpha + \tfrac{1}{2}\right) \tag{1}$$

and equation no 34 in Gambier's classification [4],

$$2w\frac{d^2w}{dz^2} - \left(\frac{dw}{dz}\right)^2 + 4w^3 - 8zw^2 + 16\alpha^2 = 0 \tag{2}$$

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which we call the Painlevé XXXIV equation (P_{XXXIV}). In equations (1) and (2), we have adopted a different scale from their canonical forms [4],

$$\frac{d^2\tilde{u}}{ds^2} = 2\tilde{u}^3 + s\tilde{u} + \beta \tag{3}$$

$$2\tilde{w}\frac{d^2\tilde{w}}{ds^2} - \left(\frac{d\tilde{w}}{ds}\right)^2 - 8\gamma\tilde{w}^3 + 2s\tilde{w}^2 + 1 = 0 \tag{4}$$

for clarity of expression of solutions. The scale transformations,

$$\begin{aligned} &s = (-4)^{1/3}z \qquad \tilde{u} = (-4)^{-1/3}u \qquad \tilde{w} = \pm 2^{-4/3}\alpha^{-1}w \\ &\beta = -\left(\alpha + \tfrac{1}{2}\right) \qquad \gamma = \mp\alpha/2 \end{aligned} \tag{5}$$

lead the canonical forms to our ones. We denote $u(\alpha)$, $P_{II}[\alpha]$ and $w(\alpha)$, $P_{XXXIV}[\alpha]$, respectively, when it is necessary to show the value of a parameter explicitly. These equations are related to each other by the Miura transformation [4],

$$w = -\frac{du}{dz} - u^2 + 2z \tag{6}$$

and its complement,

$$u = \frac{dw/dz + 4\alpha}{2w}. \tag{7}$$

Umemura *et al* also proved that P_{II} (and thus P_{XXXIV}) admits transcendental classical solutions which are expressed by the Airy function when α is an integer, rational solutions when α is a half-integer and that otherwise the solutions are non-classical.

It is known that the Painlevé equations (except for P_I) admit the Bäcklund transformations (BT) which form the affine Weyl group [8–11]. For example, BT of P_{II} is given by

$$S : S(u) = u + \frac{4\alpha}{du/dz + u^2 - 2z} \qquad S(\alpha) = -\alpha \tag{8}$$

$$T : T(u) = -u + \frac{4(\alpha+1)}{du/dz - u^2 + 2z} \qquad T(\alpha) = \alpha + 1 \tag{9}$$

and $\langle S, T\rangle$ forms the affine Weyl group of type $A_1^{(1)}$. Starting from a suitable 'seed' solution, we obtain 'higher' solutions by applying BT to it, which are expressed by rational functions in the seed solution and its derivatives.

It is interesting and important to note two points on classical solutions for the Painlevé equations. The first is that, in general, the classical solutions are located on special points in the parameter space from the viewpoint of symmetry. Namely, transcendental classical solutions are on the walls of the Weyl chambers, and rational solutions on their barycentres. The second is that the classical solutions have additional structure. Namely, it is known that some of the classical solutions admit such determinant formulae that they are expressed by a log derivative of the ratio of some determinants. Moreover, application of BT corresponds to an increment of the size of the determinant. In fact, both Airy function type and rational solutions for P_{II} and P_{XXXIV} admit such a determinant structure.

Then, does such determinant structure exist even for non-classical solutions, or are non-classical solutions so 'transcendental' that do not admit even such structure? Here we also note that such determinant structure are shown to be universal among the soliton equations due to the celebrated Sato theory [12, 13].

The purpose of this paper is to generalize the determinant formulae of the classical solutions of P_{II} and P_{XXXIV}, and show that such determinant structure is universal among the solutions of P_{II} and P_{XXXIV}. The key to this generalization is to use the technique of the Darboux transformation together with Hirota's bilinear formalism.

This paper is organized as follows. In section 2, we summarize the derivation of P_{XXXIV} and classical solutions of P_{II} and P_{XXXIV}. We illustrate our basic ideas in section 3. Our main results are presented in section 4. The proof for our results are given in section 5. Section 6 is devoted to a summary and discussions.

2. P_{XXXIV} and classical solutions of P_{II} and P_{XXXIV}

It is well known that P_{II} is derived as the similarity reduction from the modified KdV equation [7],

$$u_t + \tfrac{3}{2}u^2 u_x - \tfrac{1}{4}u_{xxx} = 0. \tag{10}$$

In fact, putting

$$u(x,t) = \frac{1}{(3t)^{1/3}}u(z) \qquad z = \frac{x}{(3t)^{1/3}} \tag{11}$$

equation (10) is reduced to equation (1). Similarly, P_{XXXIV} (2) is derived as the similarity reduction from the KdV equation,

$$w_t - \tfrac{3}{2}ww_x - \tfrac{1}{4}w_{xxx} = 0 \tag{12}$$

by putting

$$w(x,t) = \frac{1}{(3t)^{2/3}}(w(z) - 2z) \qquad z = \frac{x}{(3t)^{1/3}}. \tag{13}$$

It is known that P_{II} (1) is bilinearized to [14]

$$\left(D_z^2 - \mu\right) g\cdot f = 0 \tag{14}$$

$$\left[D_z^3 + (4z - 3\mu)D_z - 4\left(\alpha + \tfrac{1}{2}\right)\right] g\cdot f = 0 \tag{15}$$

by the dependent variable transformation,

$$u = \frac{\mathrm{d}}{\mathrm{d}z}\log\frac{g}{f} \tag{16}$$

where μ is an arbitrary function in z, and D_z is Hirota's bilinear differential operator defined by

$$D_z^n\, g\cdot f = \left(\frac{\mathrm{d}}{\mathrm{d}z} - \frac{\mathrm{d}}{\mathrm{d}z'}\right)^n g(z)f(z')\bigg|_{z'=z}. \tag{17}$$

The bilinear equations (14) and (15) are regarded as those for P_{II}, but it is also possible to derive P_{XXXIV} as follows. First we divide equations (14) and (15) by f^2. Then, putting

$$\psi = \frac{g}{f} \qquad w = 2\frac{\mathrm{d}^2}{\mathrm{d}z^2}\log f - \mu + 2z \tag{18}$$

and using the formulae [16, 17]

$$\frac{D_z g\cdot f}{f^2} = \frac{\mathrm{d}}{\mathrm{d}z}\psi \tag{19a}$$

$$\frac{D_z^2 g\cdot f}{f^2} = \left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} + w + \mu - 2z\right)\psi \tag{19b}$$

$$\frac{D_z^3 g\cdot f}{f^2} = \left[\frac{\mathrm{d}^3}{\mathrm{d}z^3} + 3(w + \mu - 2z)\frac{\mathrm{d}}{\mathrm{d}z}\right]\psi \tag{19c}$$

we obtain

$$\left(2w\frac{\mathrm{d}}{\mathrm{d}z} - \frac{\mathrm{d}w}{\mathrm{d}z} - 4\alpha\right)\psi = 0 \tag{20}$$

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} + w - 2z\right)\psi = 0. \tag{21}$$

The compatibility condition of the linear equations (20) and (21) gives $\mathrm{P}_{\mathrm{XXXIV}}[\alpha]$.

Let us discuss the determinant expressions for the classical solutions of P_{II} and $\mathrm{P}_{\mathrm{XXXIV}}$ [10, 14, 15]. In the following, the determinant with size zero should be regarded as 1.

If we choose $\mu = 2z$, then it is known that the bilinear equations (14) and (15) are satisfied by

$$g = \rho_{N+1} \qquad f = \rho_N \qquad \alpha = N \qquad N \in \mathbb{Z}_{\geqslant 0} \tag{22}$$

$$\rho_N = \begin{vmatrix} f^{(0)} & f^{(1)} & \cdots & f^{(N-1)} \\ f^{(1)} & f^{(2)} & \cdots & f^{(N)} \\ \vdots & \vdots & \ddots & \vdots \\ f^{(N-1)} & f^{(N)} & \cdots & f^{(2N-2)} \end{vmatrix} \qquad \rho_0 = 1 \tag{23}$$

where $f^{(m)} = \frac{\mathrm{d}^m}{\mathrm{d}z^m} f$ and f satisfies the Airy equation,

$$\frac{\mathrm{d}^2 f}{\mathrm{d}z^2} = 2zf. \tag{24}$$

Thus, the Airy-function-type solutions of $\mathrm{P}_{\mathrm{II}}[N]$ and $\mathrm{P}_{\mathrm{XXXIV}}[N]$ are given by

$$u = \frac{\mathrm{d}}{\mathrm{d}z}\log\frac{\rho_{N+1}}{\rho_N} \tag{25a}$$

$$w = 2\frac{\mathrm{d}^2}{\mathrm{d}z^2}\log\rho_N \tag{25b}$$

respectively.

The rational solutions are obtained for the case of $\mu = 0$, and there are two expressions for them. One is the Schur-function-type expression given by

$$g = \sigma_{N+1} \qquad f = \sigma_N \qquad \alpha = N + \tfrac{1}{2} \qquad N \in \mathbb{Z}_{\geqslant 0} \tag{26}$$

$$\sigma_N = \begin{vmatrix} q_N & q_{N+1} & \cdots & q_{2N-1} \\ q_{N-2} & q_{N-1} & \cdots & q_{2N-3} \\ \vdots & \vdots & \ddots & \vdots \\ q_{-N+2} & q_{-N+3} & \cdots & q_1 \end{vmatrix} \qquad \sigma_0 = 1 \tag{27}$$

where q_k are the polynomials in z defined by

$$\sum_{k=0}^{\infty} q_k(z)\,\xi^k = \exp\left(z\xi + \tfrac{1}{3}\xi^3\right) \qquad q_k(z) = 0 \quad \text{for } k < 0. \tag{28}$$

Another expression is given by Hankel determinant as

$$g = \kappa_{N+1} \qquad f = \kappa_N \qquad \alpha = N + \tfrac{1}{2} \qquad N \in \mathbb{Z}_{\geqslant 0} \tag{29}$$

$$\kappa_N = \begin{vmatrix} a_0 & a_1 & \cdots & a_{N-1} \\ a_1 & a_2 & \cdots & a_N \\ \vdots & \vdots & \ddots & \vdots \\ a_{N-1} & a_N & \cdots & a_{2N-2} \end{vmatrix} \qquad \kappa_0 = 1 \tag{30}$$

where a_n, $n = 0, 1, 2, \ldots$ are the polynomials in z, which are defined recursively by

$$a_n = \frac{\mathrm{d}a_{n-1}}{\mathrm{d}z} + \sum_{k=0}^{n-2} a_k a_{n-k-2} \qquad n > 0 \quad a_0 = z. \tag{31}$$

Thus, the rational solutions for $\mathrm{P_{II}}[N + \frac{1}{2}]$ and $\mathrm{P_{XXXIV}}[N + \frac{1}{2}]$ are given by

$$u = \frac{\mathrm{d}}{\mathrm{d}z} \log \frac{\sigma_{N+1}}{\sigma_N} = \frac{\mathrm{d}}{\mathrm{d}z} \log \frac{\kappa_{N+1}}{\kappa_N} \tag{32a}$$

$$w = 2z + 2\frac{\mathrm{d}^2}{\mathrm{d}z^2} \log \sigma_N = 2z + 2\frac{\mathrm{d}^2}{\mathrm{d}z^2} \log \kappa_N \tag{32b}$$

respectively.

We note that it is possible to generalize the above result for negative α by the reflection symmetry of $\mathrm{P_{II}}$ and $\mathrm{P_{XXXIV}}$,

$$u(-\alpha - 1) = -u(\alpha) \tag{33}$$

and

$$w(-\alpha) = w(\alpha) \tag{34}$$

respectively. Indeed, this symmetry is generated by ST in equations (8) and (9).

3. Darboux transformation

The technique of Darboux transformation is well developed in the soliton theory [5]. For example, in the case of the KdV equation (12), we start with its auxiliary linear problem,

$$\begin{aligned} &-\Psi_{xx} - w\Psi = \lambda\Psi \\ &\Psi_t = \Psi_{xxx} + \tfrac{3}{2} w\Psi_x + \tfrac{3}{4} w_x \Psi \end{aligned} \tag{35}$$

where λ is the spectral parameter. Then, one can show that equations (35) are covariant with respect to the Darboux transformation $\Psi \to \Psi[N]$, $w \to w[N]$ defined by

$$\Psi[N] = \frac{Wr(\Psi_1, \Psi_2, \ldots, \Psi_N, \Psi)}{Wr(\Psi_1, \Psi_2, \ldots, \Psi_N)} \tag{36}$$

$$w[N] = w + 2\frac{\partial^2}{\partial x^2} \log Wr(\Psi_1, \Psi_2, \ldots, \Psi_N) \tag{37}$$

where Ψ_k is a solution of the linear equations (35) with $\lambda = \lambda_k$, and Wr is a Wronskian with respect to the indicated functions. Thus, choosing an appropriate seed solution of the KdV equation as w, one can construct series of exact solutions by using this method. For example, starting from the solution $w = 0$, we then obtain the Wronskian expression of the N-soliton solution [6]. Although most of the expression would be somewhat formal except for the soliton and rational-type solutions or their variants, it is important that we can express quite a wide class of solutions in terms of the determinant.

The above result is recovered by the following bilinear equations:

$$\left(D_x D_t - \tfrac{1}{4} D_x^4 - \tfrac{3}{2} w D_x\right) F \cdot F = 0 \tag{38}$$

$$\left(D_x^2 + \lambda + w\right) G \cdot F = 0 \tag{39}$$

$$\left[D_x^3 - 4D_t + 3(-\lambda + w) D_x\right] G \cdot F = 0 \tag{40}$$

where $F = Wr(\Psi_1, \Psi_2, \ldots, \Psi_N)$, $G = Wr(\Psi_1, \Psi_2, \ldots, \Psi_N, \Psi)$. We omit the detail, but we can directly prove that these bilinear equations hold as the identity of the determinants from the linear equations (35). Indeed, we recover the 'usual' bilinear equations by putting $w = 0$,

$$\left(D_x D_t - \tfrac{1}{4} D_x^4\right) F \cdot F = 0 \tag{41}$$

$$\left(D_x^2 + \lambda\right) G \cdot F = 0 \tag{42}$$

$$\left(D_x^3 - 4D_t - 3\lambda D_x\right) G \cdot F = 0. \tag{43}$$

From these bilinear equations, we recover the KdV equation (12) and the auxiliary linear problem (35) by putting

$$w = 2\frac{\partial^2}{\partial x^2} \log F \qquad \Psi = \frac{G}{F}. \tag{44}$$

Now, keeping this correspondence between the Darboux transformation and the bilinear formalism in mind, let us illustrate our strategy to generalize the determinant formulae for the solutions of $\mathrm{P_{II}}$ and $\mathrm{P_{XXXIV}}$. We have the bilinear equations (14) and (15) for $\mathrm{P_{XXXIV}}$. Clearly, these bilinear equations correspond to equations (42) and (43). Thus, if we could obtain 'generalized' bilinear equations which correspond to equations (39) and (40), it might be possible to find the Darboux transformation for $\mathrm{P_{XXXIV}}$ and $\mathrm{P_{II}}$ which leaves the linear equations (20) and (21) covariant. Then we obtain a generalization of the determinant formulae.

In the next section, we present our main results.

4. Main results

In this section, we present a generalization of determinant formulae for the solutions of $\mathrm{P_{II}}$ and $\mathrm{P_{XXXIV}}$.

We start with a pair of linear equations (20) and (21). As we mentioned in section 2, these equations are compatible provided that w satisfies $\mathrm{P_{XXXIV}}[\alpha]$. In other words, equation (21) follows if w and ψ are solutions of $\mathrm{P_{XXXIV}}[\alpha]$ and equation (20), respectively. Then we have the following theorem.

Theorem 4.1. *Let w be a solution of $P_{\mathrm{XXXIV}}[\alpha]$, and ψ_0 be a solution of the linear equation (20). We define two sequences ψ_n, φ_n ($n = 0, 1, 2, \ldots$) by*

$$\varphi_0 = \frac{w}{2\psi_0} \tag{45}$$

$$\psi_n = \frac{\mathrm{d}\psi_{n-1}}{\mathrm{d}z} + \frac{w}{2\psi_0}\sum_{k=0}^{n-2} \psi_k \psi_{n-2-k} \qquad n > 0 \tag{46}$$

$$\varphi_n = \frac{\mathrm{d}\varphi_{n-1}}{\mathrm{d}z} + \psi_0 \sum_{k=0}^{n-2} \varphi_k \varphi_{n-2-k} \qquad n > 0. \tag{47}$$

We define the Hankel determinant τ_N, $N \in \mathbb{Z}$ by

$$\tau_N = \begin{cases} \det(\psi_{i+j-2})_{i,j=1,\ldots,N} & N > 0 \\ 1 & N = 0 \\ \det(\varphi_{i+j-2})_{i,j=1,\ldots-N} & N < 0. \end{cases} \tag{48}$$

Then,

$$\Psi_N = \frac{\tau_{N+1}}{\tau_N} \tag{49}$$

satisfies

$$\left[2W\frac{\mathrm{d}}{\mathrm{d}z} - \frac{\mathrm{d}W}{\mathrm{d}z} - 4(\alpha + N)\right]\Psi_N = 0 \tag{50}$$

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} + W - 2z\right)\Psi_N = 0 \tag{51}$$

where

$$W = w + 2\frac{\mathrm{d}^2}{\mathrm{d}z^2}\log\tau_N. \tag{52}$$

Thus, W satisfies $P_{\mathrm{XXXIV}}[\alpha + N]$.

A similar formula for $\mathrm{P_{II}}$ is obtained by applying the Miura transformation (6). The linear equations (20) and (21) are reduced to

$$\left(\frac{\mathrm{d}}{\mathrm{d}z} - u\right)\psi = 0. \tag{53}$$

Then we have the following theorem.

Theorem 4.2. *Let u be a solution of $P_{\mathrm{II}}[\alpha]$, and τ_N be the Hankel determinant given by equation (48), where ψ_n and φ_n are defined recursively by*

$$\psi_0 = \exp\left(\int \mathrm{d}z\, u\right) \qquad \varphi_0 = \frac{-\mathrm{d}u/\mathrm{d}z - u^2 + 2z}{2\psi_0} \tag{54}$$

$$\psi_n = \frac{\mathrm{d}\psi_{n-1}}{\mathrm{d}z} + \varphi_0\sum_{k=0}^{n-2}\psi_k\psi_{n-2-k} \qquad n > 0 \tag{55}$$

$$\varphi_n = \frac{\mathrm{d}\varphi_{n-1}}{\mathrm{d}z} + \psi_0\sum_{k=0}^{n-2}\varphi_k\varphi_{n-2-k} \qquad n > 0. \tag{56}$$

Then,

$$U = \frac{\mathrm{d}}{\mathrm{d}z}\log\frac{\tau_{N+1}}{\tau_N} \tag{57}$$

satisfies $P_{\mathrm{II}}[\alpha + N]$.

As mentioned in the previous section, theorems 4.1 and 4.2 are the direct consequence of the following proposition.

Proposition 4.3. *The following bilinear equations hold:*

$$(D_z^2 + w - 2z)\tau_{N+1}\cdot\tau_N = 0 \tag{58}$$

$$\left[D_z^3 + (3w - 2z)D_z - 4\left(\alpha + N + \tfrac{1}{2}\right)\right]\tau_{N+1}\cdot\tau_N = 0. \tag{59}$$

In fact, dividing equations (58) and (59) by τ_N^2 and using the formulae [16, 17]

$$\frac{D_z\tau_{N+1}\cdot\tau_N}{\tau_N^2} = \frac{\mathrm{d}}{\mathrm{d}z}\Psi_N \tag{60a}$$

$$\frac{D_z^2\tau_{N+1}\cdot\tau_N}{\tau_N^2} = \left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} + W - w\right)\Psi_N \tag{60b}$$

$$\frac{D_z^3\tau_{N+1}\cdot\tau_N}{\tau_N^2} = \left[\frac{\mathrm{d}^3}{\mathrm{d}z^3} + 3(W - w)\frac{\mathrm{d}}{\mathrm{d}z}\right]\Psi_N \tag{60c}$$

we obtain the linear equations (50) and (51) in theorem 4.1. Similarly, dividing equations (58) and (59) by $\tau_{N+1}\tau_N$ and using the formulae [16, 17],

$$\frac{D_z\tau_{N+1}\cdot\tau_N}{\tau_{N+1}\tau_N} = U \tag{61a}$$

$$\frac{D_z^2\tau_{N+1}\cdot\tau_N}{\tau_{N+1}\tau_N} = V + U^2 \tag{61b}$$

$$\frac{D_z^3\tau_{N+1}\cdot\tau_N}{\tau_{N+1}\tau_N} = \frac{\mathrm{d}^2U}{\mathrm{d}z^2} + 3UV + U^3 \tag{61c}$$

with

$$V = \frac{\mathrm{d}^2}{\mathrm{d}z^2}\log(\tau_{N+1}\tau_N) \tag{62}$$

we find

$$V + U^2 + w - 2z = 0 \tag{63}$$

$$\frac{\mathrm{d}^2U}{\mathrm{d}z^2} + 3UV + (3w-2z)U - 4\left(\alpha + N + \tfrac{1}{2}\right) = 0. \tag{64}$$

Eliminating V, we obtain

$$\frac{\mathrm{d}^2U}{\mathrm{d}z^2} = 2U^3 - 4zU + 4\left(\alpha + N + \tfrac{1}{2}\right). \tag{65}$$

These results imply that even non-classical solutions possess the same determinant structure as the classical solutions. Indeed, the known determinant expressions for the classical solutions are recovered as special cases. In fact, starting with a solution $w = 2z$ for $\mathrm{P}_{\mathrm{XXXIV}}\left[\frac{1}{2}\right]$, the linear equation (20) yields

$$\left(z\frac{\mathrm{d}}{\mathrm{d}z} - 1\right)\psi = 0 \tag{66}$$

from which we obtain $\psi_0 = z$ without losing generality. Then, we have from the recursion relations (45)–(47),

$$\psi_n = \frac{\mathrm{d}\psi_{n-1}}{\mathrm{d}z} + \sum_{k=0}^{n-2}\psi_k\psi_{n-2-k} \qquad n > 0 \quad \psi_0 = z \tag{67}$$

$$\varphi_n = \frac{\mathrm{d}\varphi_{n-1}}{\mathrm{d}z} + z\sum_{k=0}^{n-2}\varphi_k\varphi_{n-2-k} \qquad n > 0 \quad \varphi_0 = 1. \tag{68}$$

Thus, we have a series of rational solutions given by equations (48), (67), (68) and

$$U = \frac{\mathrm{d}}{\mathrm{d}z}\log\frac{\tau_{N+1}}{\tau_N} \tag{69a}$$

$$W = 2z + 2\frac{\mathrm{d}^2}{\mathrm{d}z^2}\log\tau_N. \tag{69b}$$

The case of $N \geqslant 0$ agrees with the Hankel determinant expression for the rational solutions discussed in section 2.

Next, noticing that $w = 0$ is a solution of $\mathrm{P}_{\mathrm{XXXIV}}[0]$, we find from equation (21) that ψ_0 is determined by

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} - 2z\right)\psi = 0 \tag{70}$$

and the recursion relation (46) with equation (45) is reduced to

$$\psi_n = \frac{\mathrm{d}\psi_{n-1}}{\mathrm{d}z}. \tag{71}$$

Thus, we have a series of solutions given by equations (48), (70), (71) and

$$U = \frac{\mathrm{d}}{\mathrm{d}z} \log \frac{\tau_{N+1}}{\tau_N} \tag{72a}$$

$$W = 2\frac{\mathrm{d}^2}{\mathrm{d}z^2} \log \tau_N \tag{72b}$$

for $N \geqslant 0$, which agrees with the Wronskian expression for the Airy-function-type solutions discussed in section 2. However, the Airy-function-type solutions are the 'singular' case in our formula. From equations (45) and (47), we find that $\varphi_n = 0$ for all n and thus $\tau_N = 0$ for $N < 0$, which does not give a meaningful result. We note that this phenomena is related to the symmetry of the τ sequence which will be mentioned in section 6, and the correct result for the $N < 0$ case is obtained by virtue of symmetry. We also note that this is the only singular case and our determinant formula works for other cases.

5. Proof of theorems

In this section, we give the proof of the theorems 4.1 and 4.2. Since we see that these theorems follow immediately from proposition 4.3, it is sufficient to prove it. We have to prove both $N > 0$ and $N < 0$ cases, but we concentrate on the former case, since the latter case is proved in a similar manner.

The bilinear equations (58) and (59) are reduced to the Plücker relations, which are quadratic identities of the determinants whose columns are shifted. Thus, we first construct differential formulae such that shifted determinants are expressed by operating some differential operator on the original determinant. For this purpose, we introduce the following notation.

Definition 5.1. *Let Y be a Young diagram $Y = (i_1, i_2, \ldots, i_h)$. Then we define an $N \times N$ determinant τ_{NY} by*

$$\tau_{NY} = \begin{vmatrix} \psi_0 & \psi_1 & \cdots & \psi_{N-h-1} & \psi_{N-h+i_h} & \cdots & \psi_{N-2+i_2} & \psi_{N-1+i_1} \\ \psi_1 & \psi_2 & \cdots & \psi_{N-h} & \psi_{N-h+1+i_h} & \cdots & \psi_{N-1+i_2} & \psi_{N+i_1} \\ \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ \psi_{N-1} & \psi_N & \cdots & \psi_{2N-h-2} & \psi_{2N-h-1+i_h} & \cdots & \psi_{2N-3+i_2} & \psi_{2N-2+i_1} \end{vmatrix}. \tag{73}$$

Then, we have the following differential formulae.

Proposition 5.2.

$$\tau_{N\square} = \frac{\mathrm{d}}{\mathrm{d}z}\tau_N \tag{74}$$

$$\tau_{N\square\square} + \tau_{N\begin{smallmatrix}\square\\\square\end{smallmatrix}} = \left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} + \frac{w}{2}\right)\tau_N \tag{75}$$

$$\tau_{N\square\square} - \tau_{N\begin{smallmatrix}\square\\\square\end{smallmatrix}} = \left(-\tfrac{1}{2}w + 2Nz\right)\tau_N \tag{76}$$

$$\tau_{N\square\square\square} + 2\tau_{N\begin{smallmatrix}\square\square\\\square\end{smallmatrix}} + \tau_{N\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}} = \left(\frac{\mathrm{d}^3}{\mathrm{d}z^3} + \frac{3}{2}w\frac{\mathrm{d}}{\mathrm{d}z} + \frac{1}{2}\frac{\mathrm{d}w}{\mathrm{d}z}\right)\tau_N \tag{77}$$

$$\tau_{N\,\ydiagram{3}} - \tau_{N\,\ydiagram{1,1,1}} = \left[\left(-\tfrac{1}{2}w + 2Nz\right)\frac{\mathrm{d}}{\mathrm{d}z} + 2\,(\alpha - 1 + N)\right]\tau_N \tag{78}$$

$$\tau_{N\,\ydiagram{3}} - \tau_{N\,\ydiagram{2,1}} + \tau_{N\,\ydiagram{1,1,1}} = \left[2z\frac{\mathrm{d}}{\mathrm{d}z} + 2N^2 + 4N(\alpha - 1) - \frac{1}{4}\frac{\mathrm{d}w}{\mathrm{d}z} - 3(\alpha - 1)\right]\tau_N. \tag{79}$$

The proof for proposition 5.2 is an important step. However, since this requires straightforward but tedious calculations, we will give it in the appendix.

Finally, we prove proposition 4.3. From the Plücker relations we have,

$$\tau_{N+1\,\ydiagram{1,1}}\tau_N - \tau_{N+1\,\ydiagram{1}}\tau_{N\,\ydiagram{1}} + \tau_{N+1}\tau_{N\,\ydiagram{2}} = 0 \tag{80}$$

$$\tau_{N+1\,\ydiagram{2,1}}\tau_N - \tau_{N+1\,\ydiagram{2}}\tau_{N\,\ydiagram{1}} + \tau_{N+1}\tau_{N\,\ydiagram{3}} = 0. \tag{81}$$

By using proposition 5.2, we find the bilinear relations (58) and (59) from equations (80) and (81), respectively, which is the desired result.

6. Summary and discussions

In this paper, we have presented determinant formulae for the solutions of $\mathrm{P_{XXXIV}}$ and $\mathrm{P_{II}}$ which are also valid for non-classical solutions by using the technique of the Darboux transformation and bilinear formalism. The solutions of $\mathrm{P_{XXXIV}}[\alpha + N]$ and $\mathrm{P_{II}}[\alpha + N]$, $N \in \mathbb{Z}$, are expressed by determinants whose entries are constructed from the solution of some linear equations. Moreover, coefficients of those linear equations include the solution of $\mathrm{P_{XXXIV}}[\alpha]$ and $\mathrm{P_{II}}[\alpha]$, respectively. We have also shown that known determinant expressions for classical solutions are recovered as special cases. This result implies that determinant structure of the classical solutions is universal among the solutions of $\mathrm{P_{II}}$ and $\mathrm{P_{XXXIV}}$.

Finally, let us discuss the relation with the Toda equation, which will be a key for generalization to other Painlevé equations.

In general, the τ function for $\mathrm{P_{II}}$ is introduced through its Hamiltonian [10],

$$H_{\mathrm{II}}(v, u, z; \alpha) = \tfrac{1}{2}v^2 + (-u^2 + 2z)v - 4\alpha u$$

by

$$H_{\mathrm{II}}(v, u, z; \alpha) = \frac{\mathrm{d}}{\mathrm{d}z}\log\tau(\alpha). \tag{82}$$

We note that we obtain $\mathrm{P_{II}}[\alpha]$ for u from the canonical equation,

$$\frac{\mathrm{d}u}{\mathrm{d}z} = \frac{\partial H_{\mathrm{II}}}{\partial v} \qquad \frac{\mathrm{d}v}{\mathrm{d}z} = -\frac{\partial H_{\mathrm{II}}}{\partial u}. \tag{83}$$

Then it can be shown that $u(\alpha)$, which is a solution of $\mathrm{P_{II}}[\alpha]$, is expressed as

$$u(\alpha) = \frac{\mathrm{d}}{\mathrm{d}z}\log\frac{\tau(\alpha + 1)}{\tau(\alpha)}. \tag{84}$$

By applying BT, we obtain a sequence of τ functions $\{\tau_N\}_{N\in\mathbb{Z}}$, where $\tau_N = \tau(\alpha+N)$. Okamoto has shown that BT of $\mathrm{P_{II}}$ is governed by the Toda equation on the level of the τ function.

Proposition 6.1. *(Okamoto [10])* τ_N *satisfies the Toda equation,*

$$\frac{\mathrm{d}^2}{\mathrm{d}z^2}\log\tau_N = c_N\frac{\tau_{N+1}\tau_{N-1}}{\tau_N^2} \tag{85}$$

where c_N *is a non-zero constant.*

It is well known that the solution of the Toda equation (85) is expressed by

$$\tau_N = \det\left(\frac{\mathrm{d}^{i+j-2}}{\mathrm{d}z^{i+j-2}} f\right)_{i,j=1,\ldots,N} \tag{86}$$

where f is an arbitrary function in z. Equation (86) is sometimes referred as 'Darboux's formula'. However, one should note that Darboux's formula (86) is valid under the condition,

$$\tau_0 = 1 \qquad \tau_1 = f \qquad N \geqslant 0. \tag{87}$$

(In practice, τ_0 can be a constant.) In [10], it is pointed out that the Wronskian expression for the Airy-function-type solutions of $\mathrm{P_{II}}$ is a consequence of Darboux's formula. We demonstrate how we could apply Darboux's formula for this case. We have a solution of $\mathrm{P_{II}}$ for $\alpha = 0$,

$$u(0) = \frac{\mathrm{d}}{\mathrm{d}z}\log\psi \qquad \frac{\mathrm{d}^2}{\mathrm{d}z^2}\psi = 2z\psi. \tag{88}$$

Therefore, we can choose the τ functions as

$$\tau(0) = \tau_0 = 1 \qquad \tau(1) = \tau_1 = \psi. \tag{89}$$

Then we have a τ sequence for the Airy-function-type solutions,

$$\ldots, \tau_{-3}, \quad \tau_{-2}, \quad \tau_{-1}, \quad \tau_0 = \tau(0) = 1, \quad \tau_1 = \psi, \quad \tau_2, \quad \tau_3, \ldots. \tag{90}$$

Now, as mentioned in section 2, we have a reflection symmetry (33) on u which implies a symmetry on the τ function,

$$\tau(-\alpha) = \tau(\alpha). \tag{91}$$

In the case of the Airy-function-type solutions, this symmetry induces a symmetry on the τ sequence as

$$\tau_{-N} = \tau_N. \tag{92}$$

By virtue of this symmetry, we see that the τ sequence is divided into two parts as

$$\ldots, \tau_3, \quad \tau_2, \quad \tau_1, \quad \tau_0 = \tau(0) = 1, \quad \tau_1 = \psi, \quad \tau_2, \quad \tau_3, \ldots. \tag{93}$$

Fortunately enough, since it can be shown that this τ sequence is governed by the Toda equation (85) with $c_N = 1$, we could apply Darboux's formula for $N > 0$ and $N < 0$ separately. We note that if we prolong the τ sequence for $N < 0$ following to the Toda equation (85) under the condition (87) without taking the symmetry into account, we have $\tau_N = 0$ for $N < 0$. This corresponds to the 'singular' phenomena mentioned in section 4.

In the case of rational solutions, the τ sequence is again separated into two parts. However, we cannot apply Darboux's formula to this case. Let us take a solution of $\mathrm{P_{II}}$ for $\alpha = \frac{1}{2}$,

$$u\left(\tfrac{1}{2}\right) = \frac{1}{z}. \tag{94}$$

We can choose the τ function as

$$\tau\left(\tfrac{1}{2}\right) = \tau_0 = 1 \qquad \tau\left(\tfrac{3}{2}\right) = \tau_1 = z \tag{95}$$

and we have a τ sequence for the rational solutions. Now the symmetry (91) implies a symmetry on the τ sequence,

$$\tau_N = \tau_{-N-1} \tag{96}$$

and thus we have,

$$\ldots, \tau_2, \quad \tau_1, \quad \tau_0, \quad \tau_0 = \tau\left(\tfrac{1}{2}\right) = 1, \quad \tau_1 = z, \quad \tau_2, \quad \tau_3, \ldots \tag{97}$$

and again the τ sequence is separated into two parts. However, in this case, these τ functions are shown to satisfy the Toda equation of the form

$$\frac{\mathrm{d}^2}{\mathrm{d}z^2}\log\tau_N = \frac{\tau_{N+1}\tau_{N-1}}{\tau_N^2} - z. \tag{98}$$

Thus, we cannot apply Darboux's formula for the rational solutions. Of course, by introducing a gauge on the τ function as

$$\sigma_N = \mathrm{e}^{z^3/6}\,\tau_N \tag{99}$$

then σ_N satisfy the Toda equation

$$\frac{\mathrm{d}^2}{\mathrm{d}z^2}\log\sigma_N = \frac{\sigma_{N+1}\sigma_{N-1}}{\sigma_N^2}. \tag{100}$$

However, now σ_0 is not a constant.

There are two points for being able to apply Darboux's formula. The first is that τ sequence should have a symmetry which is induced from the Painlevé equation itself so that the τ sequence is separated into two parts, which is necessary to apply Darboux's formula without inconsistency. The second is that the τ sequence should satisfy the Toda equation of the form (85) under the condition (87). Both conditions are satisfied for the Airy-function-type solution, but the second condition does not hold for rational solutions. Now, for solutions which correspond to generic α, the symmetry on the τ function (91) does not induce any symmetry on the τ sequence. This observation shows that it is not trivial that the generic τ function admits determinant formula, even if it satisfies the Toda equation.

Despite the unavailability of Darboux's formula, we could obtain the determinant formula for rational solutions. This is due to the determinant formula of the general solution of the Toda equation of C-type (Toda equation with the symmetry (96)) obtained in [14].

Conversely, the general determinant formula for $\mathrm{P_{II}}$ strongly implies that it is possible to construct the determinant formula for the general solution of the Toda equation in a general setting, i.e. the Toda equation admits the determinant formula for the general solution without any symmetries or gauge on a τ sequence. Then, our results might be regarded as the special case of such a general solution for the Toda equation.

Moreover, it is known that the Bäcklund transformations for the Painlevé equations (except for $\mathrm{P_I}$) are governed by various types of Toda equations [8–11]. Thus, it might be possible to also present general determinant formula for other Painlevé equations. We shall work out this problem in the next publication.

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Appendix A

Here, we give the proof for proposition 5.2.

We first prove equation (74). Notice that $\tau_{N\Box}$ is expressed by

$$\tau_{N\Box} = \begin{pmatrix} \psi_1 & \psi_2 & \cdots & \psi_N \\ \psi_2 & \psi_3 & \cdots & \psi_{N+1} \\ \vdots & \vdots & \ddots & \vdots \\ \psi_N & \psi_{N+1} & \cdots & \psi_{2N-1} \end{pmatrix} \cdot \begin{pmatrix} \Delta_{11} & \Delta_{12} & \cdots & \Delta_{1N} \\ \Delta_{21} & \Delta_{22} & \cdots & \Delta_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_{N1} & \Delta_{N2} & \cdots & \Delta_{NN} \end{pmatrix} \tag{A1}$$

where Δ_{ij} is the (i, j)-cofactor of τ_N and $A \cdot B$ denotes a standard scalar product for $N \times N$ matrices $A = (a_{ij})$ and $B = (b_{ij})$ which is defined as

$$A \cdot B = \sum_{i,j=1}^{N} a_{ij} b_{ij} = \text{trace } A^{\,t} B. \tag{A2}$$

The first matrix of equation (A1) is rewritten by using the recursion relation (46) as

$$\begin{aligned} &\begin{pmatrix} \frac{\mathrm{d}}{\mathrm{d}z}\psi_0 & \frac{\mathrm{d}}{\mathrm{d}z}\psi_1 & \cdots & \frac{\mathrm{d}}{\mathrm{d}z}\psi_{N-1} \\ \frac{\mathrm{d}}{\mathrm{d}z}\psi_1 & \frac{\mathrm{d}}{\mathrm{d}z}\psi_2 & \cdots & \frac{\mathrm{d}}{\mathrm{d}z}\psi_N \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\mathrm{d}}{\mathrm{d}z}\psi_{N-1} & \frac{\mathrm{d}}{\mathrm{d}z}\psi_N & \cdots & \frac{\mathrm{d}}{\mathrm{d}z}\psi_{2N-2} \end{pmatrix} \\ &\quad + \frac{w}{2\psi_0} \begin{pmatrix} 0 & \psi_0^2 & \cdots & \sum_{k=0}^{N-2} \psi_k \psi_{N-2-k} \\ \psi_0^2 & \psi_0\psi_1 + \psi_1\psi_0 & \cdots & \sum_{k=0}^{N-1} \psi_k \psi_{N-1-k} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{k=0}^{N-2} \psi_k \psi_{N-2-k} & \sum_{k=0}^{N-1} \psi_k \psi_{N-1-k} & \cdots & \sum_{k=0}^{2N-3} \psi_k \psi_{2N-3-k} \end{pmatrix}. \end{aligned} \tag{A3}$$

The above matrix in the second term is separated as

$$\begin{aligned} &\begin{pmatrix} 0 & 0 & \cdots & 0 \\ \psi_0^2 & \psi_1\psi_0 & \cdots & \psi_{N-1}\psi_0 \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{k=0}^{N-2} \psi_k \psi_{N-2-k} & \sum_{k=1}^{N-1} \psi_k \psi_{N-1-k} & \cdots & \sum_{k=N-1}^{2N-3} \psi_k \psi_{2N-3-k} \end{pmatrix} \\ &\quad + \begin{pmatrix} 0 & \psi_0^2 & \cdots & \sum_{k=0}^{N-2} \psi_k \psi_{N-2-k} \\ 0 & \psi_0\psi_1 & \cdots & \sum_{k=0}^{N-2} \psi_k \psi_{N-1-k} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \psi_0\psi_{N-1} & \cdots & \sum_{k=0}^{N-2} \psi_k \psi_{2N-3-k} \end{pmatrix}. \end{aligned} \tag{A4}$$

Each of these terms gives a zero contribution in equation (A1). Hence we obtain equation (74).

Next we prove equation (75). We consider

$$\tau_{N\Box\Box} + \tau_{N\genfrac{}{}{0pt}{}{\Box}{\Box}} = \begin{pmatrix} \psi_1 & \psi_2 & \cdots & \psi_{N-1} & \psi_{N+1} \\ \psi_2 & \psi_3 & \cdots & \psi_N & \psi_{N+2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \psi_N & \psi_{N+1} & \cdots & \psi_{2N-2} & \psi_{2N} \end{pmatrix} \cdot \begin{pmatrix} \Delta_{\Box 11} & \Delta_{\Box 12} & \cdots & \Delta_{\Box 1N} \\ \Delta_{\Box 21} & \Delta_{\Box 22} & \cdots & \Delta_{\Box 2N} \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_{\Box N1} & \Delta_{\Box N2} & \cdots & \Delta_{\Box NN} \end{pmatrix} \tag{A5}$$

where $\Delta_{\square ij}$ is the $(i,\, j)$-cofactor of $\tau_{N\square}$. The first matrix in the right-hand side is equal to

$$\begin{pmatrix} \frac{\mathrm{d}}{\mathrm{d}z}\psi_0 & \frac{\mathrm{d}}{\mathrm{d}z}\psi_1 & \cdots & \frac{\mathrm{d}}{\mathrm{d}z}\psi_{N-2} & \frac{\mathrm{d}}{\mathrm{d}z}\psi_N \\ \frac{\mathrm{d}}{\mathrm{d}z}\psi_1 & \frac{\mathrm{d}}{\mathrm{d}z}\psi_2 & \cdots & \frac{\mathrm{d}}{\mathrm{d}z}\psi_{N-1} & \frac{\mathrm{d}}{\mathrm{d}z}\psi_{N+1} \\ \vdots & \vdots & \ddots & \vdots & \\ \frac{\mathrm{d}}{\mathrm{d}z}\psi_{N-1} & \frac{\mathrm{d}}{\mathrm{d}z}\psi_N & \cdots & \frac{\mathrm{d}}{\mathrm{d}z}\psi_{2N-3} & \frac{\mathrm{d}}{\mathrm{d}z}\psi_{2N-1} \end{pmatrix}$$
$$+\frac{w}{2\psi_0}\begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ \psi_0^2 & \psi_1\psi_0 & \cdots & \psi_{N-2}\psi_0 & \psi_N\psi_0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \displaystyle\sum_{k=0}^{N-2}\psi_k\psi_{N-2-k} & \displaystyle\sum_{k=1}^{N-1}\psi_k\psi_{N-1-k} & \cdots & \displaystyle\sum_{k=N-2}^{2N-4}\psi_k\psi_{2N-4-k} & \displaystyle\sum_{k=N}^{2N-2}\psi_k\psi_{2N-2-k} \end{pmatrix}$$
$$+\frac{w}{2\psi_0}\begin{pmatrix} 0 & \psi_0^2 & \cdots & \sum_{k=0}^{N-3}\psi_k\psi_{N-3-k} & \sum_{k=0}^{N-1}\psi_k\psi_{N-1-k} \\ 0 & \psi_0\psi_1 & \cdots & \sum_{k=0}^{N-3}\psi_k\psi_{N-2-k} & \sum_{k=0}^{N-1}\psi_k\psi_{N-k} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \psi_0\psi_{N-1} & \cdots & \sum_{k=0}^{N-3}\psi_k\psi_{2N-4-k} & \sum_{k=0}^{N-1}\psi_k\psi_{2N-2-k} \end{pmatrix}. \tag{A6}$$

Taking the scalar product, the first and second terms give $\frac{\mathrm{d}}{\mathrm{d}z}\tau_{N\square}$ and $\frac{1}{2}w\tau_N$, respectively, and the third term vanishes. Hence we have equation (75).

Next we prove equation (76). We consider the following equality:

$$\tau_{N\square\square}-\tau_{N\begin{smallmatrix}\square\\\square\end{smallmatrix}}=\begin{pmatrix} \psi_2 & \psi_3 & \cdots & \psi_{N+1} \\ \psi_3 & \psi_4 & \cdots & \psi_{N+2} \\ \vdots & \vdots & \ddots & \vdots \\ \psi_{N+1} & \psi_{N+2} & \cdots & \psi_{2N} \end{pmatrix}\cdot\begin{pmatrix} \Delta_{11} & \Delta_{12} & \cdots & \Delta_{1N} \\ \Delta_{21} & \Delta_{22} & \cdots & \Delta_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_{N1} & \Delta_{N2} & \cdots & \Delta_{NN} \end{pmatrix}. \tag{A7}$$

The first matrix of right-hand side is rewritten as

$$\begin{pmatrix} \frac{\mathrm{d}}{\mathrm{d}z}\psi_1 & \frac{\mathrm{d}}{\mathrm{d}z}\psi_2 & \cdots & \frac{\mathrm{d}}{\mathrm{d}z}\psi_N \\ \frac{\mathrm{d}}{\mathrm{d}z}\psi_2 & \frac{\mathrm{d}}{\mathrm{d}z}\psi_3 & \cdots & \frac{\mathrm{d}}{\mathrm{d}z}\psi_{N+1} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\mathrm{d}}{\mathrm{d}z}\psi_N & \frac{\mathrm{d}}{\mathrm{d}z}\psi_{N+1} & \cdots & \frac{\mathrm{d}}{\mathrm{d}z}\psi_{2N-1} \end{pmatrix}$$
$$+\frac{w}{2\psi_0}\begin{pmatrix} 0 & 0 & \cdots & 0 \\ \psi_1\psi_0 & \psi_2\psi_0 & \cdots & \psi_N\psi_0 \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{k=1}^{N-1}\psi_k\psi_{N-1-k} & \sum_{k=2}^{N}\psi_k\psi_{N-k} & \cdots & \sum_{k=N}^{2N-2}\psi_k\psi_{2N-2-k} \end{pmatrix}$$
$$+\frac{w}{2\psi_0}\begin{pmatrix} \psi_0^2 & \psi_0\psi_1+\psi_1\psi_0 & \cdots & \sum_{k=0}^{N-1}\psi_k\psi_{N-1-k} \\ \psi_0\psi_1 & \psi_0\psi_2+\psi_1\psi_1 & \cdots & \sum_{k=0}^{N-1}\psi_k\psi_{N-k} \\ \vdots & \vdots & \ddots & \vdots \\ \psi_0\psi_{N-1} & \psi_0\psi_N+\psi_1\psi_{N-1} & \cdots & \sum_{k=0}^{N-1}\psi_k\psi_{2N-2-k} \end{pmatrix}. \tag{A8}$$

Here we note that ψ_n also satisfy

$$\frac{\mathrm{d}\psi_n}{\mathrm{d}z}=(2z-w)\psi_{n-1}+2(n-1)\psi_{n-2}-\frac{\mathrm{d}w/\mathrm{d}z-4\alpha}{4\psi_0}\sum_{k=0}^{n-2}\psi_k\psi_{n-2-k} \tag{A9}$$

which is proved by induction from equations (20), (21) and (46). The first term of the right-hand side of equation (A8) is rewritten using equation (A9) as

$$
\begin{aligned}
&(2z-w)\begin{pmatrix} \psi_0 & \psi_1 & \cdots & \psi_{N-1} \\ \psi_1 & \psi_2 & \cdots & \psi_N \\ \vdots & \vdots & \ddots & \vdots \\ \psi_{N-1} & \psi_N & \cdots & \psi_{2N-2} \end{pmatrix} \\
&\quad + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 2\psi_0 & 2\psi_1 & \cdots & 2\psi_{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ 2(N-1)\psi_{N-2} & 2(N-1)\psi_{N-1} & \cdots & 2(N-1)\psi_{2N-3} \end{pmatrix} \\
&\quad + \begin{pmatrix} 0 & 2\psi_0 & \cdots & 2(N-1)\psi_{N-2} \\ 0 & 2\psi_1 & \cdots & 2(N-1)\psi_{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 2\psi_{N-1} & \cdots & 2(N-1)\psi_{2N-3} \end{pmatrix} \\
&\quad - \frac{\mathrm{d}w/\mathrm{d}z - 4\alpha}{4\psi_0} \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \psi_0^2 & \psi_1\psi_0 & \cdots & \psi_{N-1}\psi_0 \\ \vdots & \vdots & \ddots & \vdots \\ \displaystyle\sum_{k=0}^{N-2} \psi_k\psi_{N-2-k} & \displaystyle\sum_{k=1}^{N-1} \psi_k\psi_{N-1-k} & \cdots & \displaystyle\sum_{k=N-1}^{2N-3} \psi_k\psi_{2N-3-k} \end{pmatrix} \\
&\quad - \frac{\mathrm{d}w/\mathrm{d}z - 4\alpha}{4\psi_0} \begin{pmatrix} 0 & \psi_0^2 & \cdots & \sum_{k=0}^{N-2} \psi_k\psi_{N-2-k} \\ 0 & \psi_0\psi_1 & \cdots & \sum_{k=0}^{N-2} \psi_k\psi_{N-1-k} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \psi_0\psi_{N-1} & \cdots & \sum_{k=0}^{N-2} \psi_k\psi_{2N-3-k} \end{pmatrix}. \qquad \text{(A10)}
\end{aligned}
$$

Applying the scalar product on these terms, we obtain equation (76). We obtain equations (77)–(79) by similar calculations.

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